Decoherence in time evolution of bound entanglement

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We study a dynamic process of disentanglement by considering the time evolution of bound entanglement for a quantum open system, two qutrits coupling to a common environment. Here, the initial quantum correlations of the two qutrits are characterized by the bound entanglement. In order to show the universality of the role of environment on bound entanglement, both bosonic and spin environments are considered. We found that the bound entanglement displays collapses and revivals, and it can be stable against small temperature and time change. The thermal fluctuation effects on bound entanglement are also considered.

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I. INTRODUCTION

Entanglement [1], as an essential feature of quantum mechanics, helps us to distinguish the classical and quantum nature of matter world. It has become a key ingredient in quantum information processing, such as quantum computing, quantum teleportation and quantum cryptography [2]-[4]. On the other hand, generally a realistic system is surrounded by an environment. Thus the effects of the quantum decoherence such as quantum dephasing on quantum entanglement should be considered for quantum opene systems. It is reasonable that when we study quantum effects induced by entanglement, the two-particle system should hold phase relations between the components of the entangled states. Thus perceivably, due to the interactions with environment, we can expect that the dephasing of two-particle system can demonstrate some exotic properties.

Recently, Yu and Eberly [5] showed that two entangled qubits became completely disentangled in a finite time under the influence of pure vacuum noise. Surprisingly, they found that the behaviors of local decoherence is different from the spontaneous disentanglement. The decoherence effects take an infinite time evolution under the influence of vacuum noise while the entanglement displays a "sudden death" in a finite time. In their investigations and other studies on disentanglement in open quantum systems [6] [7], only qubit systems are considered. Here, the disentanglement process is characterized by time evolution of the concurrence [8]. It is well-known that the concurrence is not available in the higher-dimensional systems.

For the systems with spins larger than 1/2, one can use the positive partial transpose (PPT) method [9] to study disentanglement. For the mixed states of two spin halves and (1/2 ,1) mixed spins, the PPT method can fully characterize entanglement. However, in the case of two qutrits and even larger spins, one only know that if a state does not have PPT, the state must be entangled. In other words, one may use the method to witness entanglement. Actually, in the case of higher dimension, there are two qualitatively different types of entanglement [10], free entanglement (FE) which corresponds to the states without a PPT, and bound entanglement (BE) corresponding to the entangled states, however, with a PPT. The BE is an intrinsic property and cannot be distilled to a singlet form, thus it cannot be used alone for quantum communication. Nevertheless, the BE can be activated and then contribute to quantum communication [11]. A formal entanglement-energy analogy [12] implies that the bound entanglement is like the energy of a system confined in a shallow potential well. If we add a small amount of extra energy, behaving as a perturbation, to the system, its energy can be deliberated. The existence of bound entangled states reveals a transparent form of irreversibility in entanglement processing [13].

In this paper, we consider an open composite system, a two-qutrit system commonly coupled to an environment, and study a type of dynamical process of disentanglement, where the two qutrits are initially prepared in a bound entangled state. We would like to reveal that different environment gives different dynamics of entanglement. Firstly, a bosonic heat bath is considered. We remark that this modeling of environment is universal [14, 15] in the sense that any environment weakly coupled to a system can be approximated by a collection of harmonic oscillators. Secondly, we consider a spin environment consisting of spin halves which can be considered as a fermionic environment. We let two types of bound

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entangled states being initial state of the two qutrits in order to find the different properties of bound entangled state during the quantum dephasing. And initially the environments are assumed to be at thermal equilibrium states, which helps us to find effects of the thermal fluctuation on dynamics of quantum entanglement.

This paper is organized as follows. In Sec. II, we consider the bosonic environment and give the analytical results of FE and BE. we numerically study the BE to illustrate the details of the dynamics of entanglement. In Sec. III, the two qutrits are coupled to a spin environment. Also the analytical and numerical results are given to show the effects of coupling strength, temperature and energy spectrum structure on the dynamical behaviors of entanglement. The conclusion is given in Sec. IV.

II. BOUND ENTANGLEMENT IN A BOSONIC ENVIRONMENT

We start with a well known model of the pure dephasing [15, 16], where two qutrits interact with the environment, which is modelled as a heat bath with many harmonic oscillators of frequency ω_j. The model Hamiltonian reads

$$H = \sum_j^L H_j = \sum_j^L [\hbar\omega_j b_j^\dagger b_j + g(b_j^\dagger + b_j)(S_{1z} + S_{2z})], \quad (1)$$

where $b_j^\dagger$ and b_j are creation and annihilation operators, respectively, S_{1z} and S_{2z} are z components of two spin-1 operators, and g denotes the coupling strength between the spins and the heat bath.

In order to study the dynamical process of entanglement in our system, it is convenient for us to study the time evolution in the interaction picture. Here,

$$H_0 = \sum_j \hbar\omega_j b_j^\dagger b_j \quad (2)$$

is the free Hamiltonian, and the interaction Hamiltonian

$$H_I = \sum_j g(b_j^\dagger + b_j)(S_{1z} + S_{2z}). \quad (3)$$

Then, through the Wei-Norman method [17], the time evolution operator in the interaction picture is factorized as,

$$U(t) = \prod_j e^{i\Phi_j(t)S_z^2} D\left[z_j(t)S_z\right], \quad (4)$$

where $S_z = S_{1z} + S_{2z}$,

$$\begin{aligned} \Phi_j(t) &= \frac{g^2}{\hbar^2\omega_j^2}\left(\omega_j t - \sin\omega_j t\right), & (5) \\ z_j(t) &= \frac{g}{\hbar\omega_j}\left(1 - e^{i\omega_j t}\right) & (6) \end{aligned}$$

and $D(z_j S_z) = \exp\left[\left(z_j b_j^\dagger - z_j^* b_j\right) S_Z\right]$ is the displacement operator.

Before discussing the dynamical process of entanglement, we introduce two quantities to quantitatively study entanglement. One is the negativity [18], which can be used to study FE. For a state ρ, negativity is defined in terms of the trace norm of the partial transposed matrix

$$\mathcal{N}(\rho) = \frac{\|\rho^{T_1}\|_1 - 1}{2}, \quad (7)$$

where T_1 denotes the partial transpose with respect to the first subsystem. If $\mathcal{N} > 0$, then the two-spin state is free entangled. As an entanglement measure, the negativity is operational and easy to compute, and it has been used to characterize entanglement in large spin system very well [19]- [21].

In order to characterize BE, one can use the so-called realignment criterion (cross-norm criterion) which proved to be very efficient [22]. The operation of realignment on the density matrix is just as $(\rho^R)_{ij,kl} = \rho_{ik,jl}$. A separate state ρ always satisfies $||\rho^R|| \leq 1$. Thus, a quantity for the BE can be defined as

$$\mathcal{R}(\rho) = \max\left\{0, ||\rho^R|| - 1\right\}. \quad (8)$$

We call this the witness quantity. Only when $\mathcal{R}(\rho) > 0$ and $\mathcal{N}(\rho)$ =0, the state is bound entangled.

A. Horodecki's Bound entangled state

In the following discussions, we consider the dynamical evolution process of the two-spin 1 system, deriven by the Hamiltonian (1) with the initial state being in the Horodecki's bound entangled state [11].

1. Analytical results

The bound entangled state reads [11]:

$$\begin{aligned} \rho_a(0) &= \frac{2}{7}\mathbf{P}_+ + \frac{a}{7}\varrho_+ + \frac{5-a}{7}\varrho_-, \\ 2 &\leq a \leq 5, \end{aligned} \quad (9)$$

where

$$\begin{aligned} \mathbf{P}_+ &= |\Psi_+\rangle\langle\Psi_+|, |\Psi_+\rangle = \\ &\frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle), & (10) \\ \varrho_+ &= \frac{1}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|), \\ \varrho_- &= \frac{1}{3}(|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|). & (11) \end{aligned}$$

where $|m_1 m_2\rangle, (m_1, m_2 = 0, 1, 2)$ are the eigenvectors of $S_z = S_{1z} + S_{2z}$, with the corresponding eigenvalues $m_1 + m_2 - 2$, respectively.

In Ref. [10], Horodecki demonstrated that

$$\rho_a \text{ is } \begin{cases} \text{separable for } 2 \le a \le 3, \\ \text{bound entangled for } 3 < a \le 4, \\ \text{free entangled for } 4 < a \le 5. \end{cases} \tag{12}$$

And the density matrix for the initial state of the total system is a simple direct product

$$\rho_{\text{tot}}(0) = \rho_a \otimes \rho_E, \tag{13}$$

where ρ_E is the density matrix of environment.

Driven by the time evolution operator (4), the system will evolve from the bound entangled state ρ_a into the state described by

$$\begin{aligned} \rho_{1,2}(t) &= \mathrm{Tr}_E\left[U(t)\,\rho_{\text{tot}}(0)\,U^\dagger(t)\right] \\ &= \frac{2}{21}\big[\left(|00\rangle\langle 00| + |11\rangle\langle 11| + |22\rangle\langle 22|\right) \\ &\quad + \left(F_1(t)\,|00\rangle\langle 11| + \text{H.c.}\right) \\ &\quad + \left(F_2(t)\,|22\rangle\langle 11| + \text{H.c.}\right) \\ &\quad + \left(F_3(t)\,|00\rangle\langle 22| + \text{H.c}\right)\big] \\ &\quad + \frac{a}{7}\varrho_+ + \frac{5-a}{7}\varrho_-. \end{aligned} \tag{14}$$

where

$$\begin{aligned} F_1(t) &= \mathrm{Tr}_E\left[\rho_E U_1^\dagger(t)\,U_0(t)\right] \\ F_2(t) &= \mathrm{Tr}_E\left[\rho_E U_1^\dagger(t)\,U_2(t)\right] \\ F_3(t) &= \mathrm{Tr}_E\left[\rho_E U_2^\dagger(t)\,U_0(t)\right] \end{aligned} \tag{15}$$

are *decoherence factors*[15]. The unitary operators $U_0(t)$, $U_1(t)$, and $U_2(t)$ are derived from Eq. (4) just by replacing operator $S_z = S_{1z} + S_{2z}$ with numbers $-2, 0$ and 2, respectively.

From the reduced density matrix (14), the realigned matrix becomes

$$\begin{aligned} (\rho_{12}(t))^R &= \frac{1}{21}\left(A_{3\times 3} \oplus B_{2\times 2}^{(1)} \oplus B_{2\times 2}^{(2)} \oplus B_{2\times 2}^{(3)}\right), \\ A_{3\times 3} &= \begin{pmatrix} 2 & a & 5-a \\ 5-a & 2 & a \\ a & 5-a & 2 \end{pmatrix}, \\ B_{2\times 2}^{(k)} &= \begin{pmatrix} 2F_k & 0 \\ 0 & 2F_k^* \end{pmatrix} (k = 1, 2, 3). \end{aligned} \tag{16}$$

Then, the witness quantity $\mathcal{R}$ is obtained as

$$\begin{aligned} \mathcal{R}(\rho_{1,2}) &= \max\left\{||\rho_{1,2}^R(t)|| - 1, 0\right\} \\ &= \frac{2}{21}\max\big\{\sqrt{3a^2 - 15a + 19} \\ &\quad + 2\left(|F_1| + |F_2| + |F_3|\right) - 7, 0\big\}. \end{aligned} \tag{17}$$

As mentioned above, the positive witness quantity can quantify the nontrivial BE only when the negativity vanishes. Thus, we need to calculate the time evolution of negativity.

We first make the partial transpose of ρ_{12} with respect to the second system and obtain

$$\begin{aligned} (\rho_{12}(t))^{T_2} &= \frac{1}{21}\left(C_{3\times 3} \oplus D_{2\times 2}^{(1)} \oplus D_{2\times 2}^{(2)} \oplus D_{2\times 2}^{(3)}\right), \\ C_{3\times 3} &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \\ D_{2\times 2}^{(k)} &= \begin{pmatrix} a & 2F_k \\ 2F_k^* & 5-a \end{pmatrix} (k = 1, 2, 3). \end{aligned} \tag{18}$$

Then, from the above equation, we immediately obtain the negativity

$$\mathcal{N}(\rho_{1,2}) = \frac{1}{42}\sum_{k=1}^{3}\max\left\{0, \sqrt{(2a-5)^2 + 16\left|F_k\right|^2} - 5\right\}. \tag{19}$$

Thus, we have obtained analytical expressions of witness quantity $\mathcal{R}$ and negativity $\mathcal{N}$ in terms of the three decoherence factors. It is natural to see that if the decoherence factors are zero, namely, the completely decoherence occurs, from Eqs. (17) and (19), we have $\mathcal{R} = \mathcal{N} = 0$. From Eq. (19), we can also see that in the region $3 < a \le 4$, negativity always gives zero at any time since $|F_k| \le 1$.

From the above discussions, once we know the decoherence factors, the two quantities $\mathcal{R}$ and $\mathcal{N}$ for detecting entanglement can be determined. So, we are left to obtain these decoherence factors. It is well known that high temperature may enhance the decoherence, thus it is reasonable to choose a thermal equilibrium state as the initial state of the heat bath, which is described by the density matrix

$$\begin{aligned} \rho_E &= \prod_j \rho_{Ej} = \prod_j \frac{e^{-\beta\hbar\omega_j b_j^\dagger b_j}}{\mathrm{Tr}\left[e^{-\beta\hbar\omega_j b_j^\dagger b_j}\right]} \\ &= \prod_j \left(1 - e^{-\beta\hbar\omega_j}\right) e^{-\beta\hbar\omega_j b_j^\dagger b_j}, \end{aligned} \tag{20}$$

where $\beta = 1/k_B T$, k_B is the Boltzmann's constant, and we choose $k_B = 1$ for simplicity in the following.

For the bosonic environment we calculate the decoherence factors in the coherent-state representation. The P-representation for the thermal state is given by

$$\rho_E(0) = \prod_j \rho_{Ej} = \prod_j \int \rho_j(\alpha)\,|\alpha\rangle\langle\alpha|\,d^2\alpha, \tag{21}$$

$$\rho_j(\alpha) = \frac{1}{\pi\langle n_j\rangle}\exp\left(-\frac{|\alpha|^2}{\langle n_j\rangle}\right), \tag{22}$$

where $\langle n_j\rangle = \left(e^{\beta\hbar\omega_j} - 1\right)^{-1}$ is the thermal excitation number of harmonic oscillators. From Eq. (15) and using the P-representation, one obtains the modulus of the

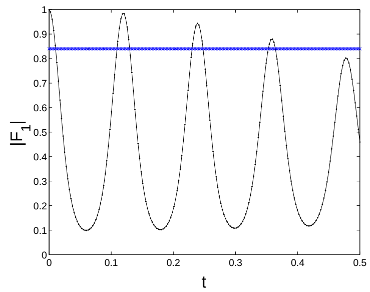

FIG. 1: The modulus of the decoherence factor $|F_1(t)|$ versus time with $g=2$. The frequencies of heat bath are chosen randomly in a region $\omega_j \in [50,55]$. The size of the bath is $L=200$ and the system is at the temperature $T=1$. The horizontal line in the figure corresponds to the threshold value $|F_1|=0.8398$.

decoherence factors [23]

$$\begin{aligned} |F_1(t)| &= |F_2(t)| = \prod_j |\mathrm{Tr}_{E_j}[\rho_{E_j} D(2z_j)] e^{i4\Phi_j}| \\ &= \prod_j e^{-2|z_j|^2(2\langle n_j\rangle+1)} \\ &= \exp(\sum_j \frac{-8g^2}{\hbar^2\omega_j^2}(2\langle n_j\rangle+1)\sin^2\left(\frac{\omega_j t}{2}\right)) \quad (23) \\ |F_3(t)| &= \prod_j \mathrm{Tr}_j\left(\rho_{Bj} D(-4z_j)\right) = |F_1(t)|^4. \quad (24) \end{aligned}$$

As expected, the above three quantities are smaller than or equal to unity. Now, we study the decoherence of BE, and choose parameter $a=4$ in the bound entangled state in the following discussions. This choice of parameter maximize the quantity $\mathcal{R}$. Then, Eq. (17) simplifies to

$$\mathcal{R}(\rho_{1,2}) = \frac{2}{21}\max\left\{0, 2\left(2|F_1| + |F_1|^4\right) + \sqrt{7} - 7\right\}. \quad (25)$$

Then, we find that the dynamic properties of BE is thus directly related to the one single decoherence factors $|F_1(t)|$. By numerical calculation, one obtains the threshold point of

$$|F_1| \approx 0.839829, \quad (26)$$

before which the quantity $\mathcal{R}$ is larger than zero, implying that the state is a bound entangled state. In Fig. 1, we numerically show the modulus $|F_1(t)|$ versus time. The frequencies ω_j are chosen randomly in a region $\omega_j \in [50,55]$. Then, the modulus $|F_1(t)|$ oscillates with time and periodically crosses the horizontal line corresponding to the threshold value $F_1 = 0.8398$. Obviously, the witness quantity displays discontinuous behavior and below the line it becomes zero.

In the following we consider some special cases of the energy distribution in the environment and find that the decoherence factors decay as a Gaussian or a exponential form with time, and consequently we know the time behaviors of the BE.

i) Let us choose a certain type of distribution of ω_j in the region $[0,\omega]$, where ω is an arbitrary value larger than zero. We do not care about the exact form of the distribution, however it can be achieved for us to choose a sufficiently small cutoff frequency ω_{j_c} to make sure that at a finite time, the decoherence factor ($\hbar = 1$)

$$\begin{aligned} |F_1(t)| &\leq \exp\sum_j^{j_c}\left[\frac{-8g^2}{\omega_j^2}\sin^2\left(\frac{\omega_j t}{2}\right)(2\langle n_j\rangle+1)\right] \\ &\approx \exp\left[-2g^2\sum_j^{j_c}(2\langle n_j\rangle+1)t^2\right] = e^{-\gamma t^2}, \end{aligned} \quad (27)$$

where

$$\gamma = 2g^2\sum_j^{j_c}(2\langle n_j\rangle+1). \quad (28)$$

It can be seen that the decoherence factor displays a Gaussian decay with time. Moreover one may observe that the decay parameter γ increases at high temperature since $\langle n_j\rangle$ is a monotonically increasing function of temperature T, and enlarging the strength g can also increase γ. Substituting Eq. (27) to (25) leads to

$$\mathcal{R}(\rho_{1,2}) = \frac{2}{21}\max\left\{0, 2\left(2e^{-\gamma t^2} + e^{-4\gamma t^2}\right) + \sqrt{7} - 7\right\}, \quad (29)$$

and it will decay to zero in a fixed time t_0, which can be determined from Eq. (26)

$$t_0 = 0.4176/\sqrt{\gamma}. \quad (30)$$

When the evolution time is larger than the threshold value t_0, the BE suddenly vanishes.

ii) If we choose some continuous spectrum, the sum in the decoherence factors (we assume $\hbar = 1$) becomes

$$\ln|F_1(t)| = -\sum_j\left[\frac{8g_j^2}{\omega_j^2}\sin^2\left(\frac{\omega_j t}{2}\right)\right], \quad (31)$$

where $g_j = g\sqrt{(2\langle n_j\rangle+1)}$. Assume a spectrum distribution $\rho(\omega_j)$, the above equation becomes

$$\ln|F_1(t)| = -\int_0^\infty \frac{8\rho(\omega_j)g_j^2}{\omega_j^2}\sin^2\frac{\omega_j t}{2}d\omega_j. \quad (32)$$

For some concrete spectrum distributions, interesting circumstances may arise. For instance, when $\rho(\omega_j) = \gamma/(2\pi g_j^2)$ the integral converges to a negative number proportional to time t, precisely, $|F_1(t)| = \exp(-\gamma t)$, $|F_3(t)| = \exp(-4\gamma t)$. Thus, in this case, the reasonable

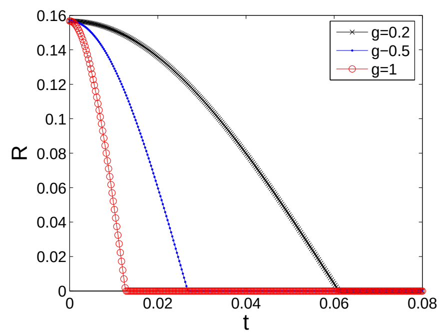

FIG. 2: $\mathcal{R}$ versus time with different coupling parameter g. The frequencies of heat bath are chosen randomly in a low region $\omega_j \in [0, 5]$. The size of bath $L = 200$ and the system is at a finite temperature $T = 1$.

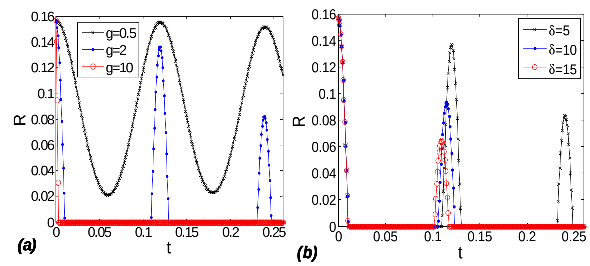

FIG. 3: **(a)** $\mathcal{R}$ versus time with different coupling parameter g. The frequencies of heat bath are chosen randomly in a higher region $\omega_j \in [50, 55]$. The size of the bath $L = 200$ and the the
freq ling
$g =$

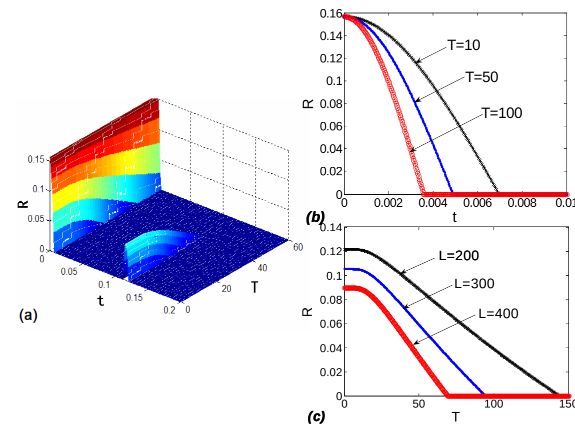

FIG. 4: **(a)** Three dimensional (3D) diagram of $\mathcal{R}$ versus time and temperature, with $L = 200$ and $g = 3$. The region of frequencies is $\omega_j \in [50, 58]$. **(b)** Quantity $\mathcal{R}$ versus time at different temperatures. **(c)** $\mathcal{R}$ versus temperature at a fixed time $t = 0.003$ for different sizes.

assumption on the energy distribution brings us a exponential decay of decoherence factor and entanglement with time.

iii) Now we will choose another more general distribution $\rho(\omega)$. Assume that all the coefficients g_j are equal: $g_j = G$. If the frequencies lie within an interval $[\omega_1, \omega_2]$ and the distribution is homogeneous, we have $\rho(\omega_k) = N/(\omega_2 - \omega_1)$, thus [24]

$$
\begin{aligned}
&\ln |F_1(t)| \\
&= -\sum_j \left[\frac{8g_j^2}{\omega_j^2} \sin^2 \frac{\omega_j t}{2} \right] \\
&= -\int_{\frac{\omega_1}{2}}^{\frac{\omega_2}{2}} \sin^2(\omega_k t) \frac{2G^2 \rho(\omega_k)}{\omega_k^2} d\omega_k \\
&= \frac{-2G^2 N}{\omega_2 - \omega_1} \int_{\frac{\omega_1}{2}}^{\frac{\omega_2}{2}} \frac{\sin^2 \omega_k t}{\omega_k^2} d\omega_k \\
&\leq \frac{-2G^2 N}{\omega_2 - \omega_1} \frac{4}{\omega_2^2} \int_{\frac{\omega_1}{2}}^{\frac{\omega_2}{2}} \sin^2(\omega_k t)\, d\omega_k \\
&= \frac{-2G^2 N}{\omega_2^2} \left[1 - \frac{2\cos\left(\frac{\omega_2+\omega_1}{2} t\right) \sin\left(\frac{\omega_2-\omega_1}{2} t\right)}{(\omega_2 - \omega_1)\, t} \right]. \quad (33)
\end{aligned}
$$

By substituting the above equation into the Eq. (25), we see that when the environment has sufficiently large size L, the decoherence factor and the quantity $\mathcal{R}$ will decay with time rapidly.

2. *Numerical results*

Next, we resort to numerical calculation to test the above analysis and show more dynamic behaviors of entanglement.

In Fig. 2, we choose a random distribution of the environment energy ω_j over a finite region $[0, \omega]$, and give the time behaviors of the quantity $\mathcal{R}$. A Gaussian decay is exhibited, and this is consistent with the previous analysis (Eq. (27)). Of course, larger g accelerates the decay process of the BE. When we change other parameters, such as the width of frequencies and the temperature, the quantity $\mathcal{R}$ displays similar behaviors as long as the region of ω_j begins from zero. This implies that the harmonic oscillators with lower energies in the heat bath determine the behaviors of the BE. .

Instead of choosing the low energies of the harmonic oscillators in environment, we consider a random distribution of ω_j in a higher frequency region $[50, 50 + \delta]$, where δ is the width of the distribution. The numerical results are shown in Fig. 3. For small values of the coupling constants, as shown in Fig. 3 (a), the BE displays oscillations with time. For $g = 2$, we observe collapses and revivals of the BE. The revivals result from the revivals of the decoherence factors. When the coupling

strength is strong enough, the BE decays rapidly to zero without revivals.

Fig. 3(b) presents numerical results for different frequency widths. Consider an extreme case $\delta = 0$, that is, only one frequency is taken into account. In this case, the BE displays periodic collapse and revivals for large g, and the revival amplitude of $\mathcal{R}$ is one. From the figure, we see that when the frequency width increases, the revival amplitude decreases. Increasing the width δ means that the harmonic oscillators in the heat bath own much more different frequencies and the entanglement revival will be suppressed.

Now we consider the thermal effect on the BE in Fig. 4. The subfigures (a) and (b) show that the thermal fluctuation can destroy entanglement and accelerate the decaying process. With the joint effect of thermal fluctuation and strong coupling g, entanglement vanishes in a finite time without reviving. In Fig. 4(c), we can see that with the temperature increasing the entanglement decreases to zero, and enlarging the size of heat bath can suppress the BE and accelerate the decay.

B. Second bound entangled state

We choose another 3×3 bound entangled state as the initial state of the two qutrits which was introduced by Bennett et al. [25] from the unextendible product bases:

$$\begin{aligned} |\phi_0\rangle &= \frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle), \\ |\phi_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle, \\ |\phi_2\rangle &= \frac{1}{\sqrt{2}}|2\rangle(|1\rangle - |2\rangle), \\ |\phi_3\rangle &= \frac{1}{\sqrt{2}}(|1\rangle) - |2\rangle)|0\rangle, \\ |\phi_4\rangle &= \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle), \end{aligned} \tag{34}$$

from which the density matrix could be expressed as

$$\rho = \frac{1}{4}(I_{9\times 9} - \sum_{j=0}^{4} |\phi_j\rangle\langle\phi_j|), \tag{35}$$

In this case, the dynamics of entanglement are determined by six decoherence factors, and analytical results are not available. We numerically calculate BE and FE, and the results are shown in Fig. 5.

In order to compare BE and FE, we have numerically given the time behaviors of both quantity $\mathcal{R}$ and negativity $\mathcal{N}$. We choose a higher frequency region $\omega_j \in [50, 50+\delta]$, which will induce some interesting properties of BE and FE. In Fig. 5 (a), we see that the negativity can be nonzero, in contrast with the first BE given by Horodecki et al. The nonzero negativity implies that

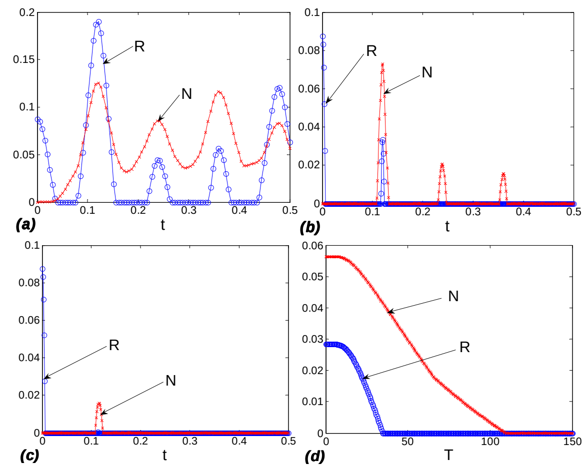

FIG. 5: We consider the BE and FE together. And in the four subfigures the blue lines with circle markers correspond to $\mathcal{R}$ and the red lines with $\times$ markers denote $\mathcal{N}$. **(a)** $\mathcal{R}$ and $\mathcal{N}$ versus time with coupling $g = 1$, the size of environment $L = 300$, and the system is at temperature $T = 10$. The frequencies of heat bath are chosen randomly in the region $\omega_j \in [50, 55]$. **(b)** Only changing the coupling to a larger one $g = 5$, and the other parameters are the same as subfigure **(a)**. In the figure **(c)**, parameter $\delta = 9$, the other parameters are the same as subfigure **(b)**. In subfigure **(d)**, it shows $\mathcal{R}$ versus temperature at $t = 0.005$ and $\mathcal{N}$ versus temperature at $t = 0.115$. $g = 1$ and $L = 300$.

the state is free entangled. The state keeps bound entangled for a short time, and then, the state is free entangled. When the coupling g becomes stronger, as shown in Fig. 5 (b), after some revivals, both quantities become zero.

In Fig. 5 (c), we extend the width of frequency region to $\delta = 9$. Expanding the energy region makes the particles in heat bath own more chances to take different energy. And this will prevent the revival of entanglement. In Fig. 5(d), we show the behaviors of $\mathcal{R}$ and $\mathcal{N}$ against temperature for a fixed time. With the increase of temperature, quantity $\mathcal{R}$ and negativity $\mathcal{N}$ decrease gradually, and at last the thermal fluctuation destroys the entanglement completely.

III. BOUND ENTANGLEMENT IN A SPIN ENVIRONMENT

To show the universality of the influence of environment on the time evolution of BE, we need to use difference modeling of environment. Here, we consider an environment consisting of N spin halves. The corresponding model Hamiltonian reads [26]

$$H = \frac{g}{2}(S_{1z} + S_{2z}) \otimes \sum_{k=1}^{L} \omega_k \sigma_z^{(k)}, \tag{36}$$

where $\sigma_z^{(k)}$ denotes the z-component of the Pauli vector, and g denotes the coupling strength between central spins and environment. We notice that the above model has been considered by Zurek [26] as a solvable model of decoherence.

The time evolution operator can be expressed as:

$$U(t)=\prod_{k=1}^{L}\exp(-it\hat{\Lambda}\omega_k\sigma_z^{(k)}), \tag{37}$$

where we define a special operator-valued parameter

$$\hat{\Lambda}=\frac{g}{2}\left(S_{1z}+S_{2z}\right). \tag{38}$$

A. Horodecki's Bound entangled state

In a similar vein as the discussions of entanglement in the bosonic environment, we first study the disentanglement of Horodecki's bound entangled state and give the analytical results.

1. Analytical results

Let us consider the whole system initially starts from a product state

$$\rho_{tot}(0)=\rho_a\otimes\rho_E,$$

where the initial state of the two qutrits ρ_a is a mixed BE state represented in Eq. (9), and ρ_E denotes the initial state of the environment which is assumed to be a thermal state described by the density matrix

$$\rho_E=\prod_{k=1}^{L}\frac{e^{\beta\omega_k\sigma_z^{(k)}}}{2\cosh(\beta\omega_k)}. \tag{39}$$

Then the reduce density matrix at time t can be given by the same matrix as Eq. (14), and now the three decohrence factors in this spin environment can be obtained as

$$\begin{aligned}|F_1(t)| &= |F_2(t)|=\prod_{k=1}^{L}|F_{1,k}| \\ &= \prod_{k=1}^{L}\sqrt{1-\frac{\sin^2(gt\omega_k)}{\cosh^2(\beta\omega_k)}}, \end{aligned}\tag{40}$$

$$\begin{aligned}|F_3(t)| &= \prod_{k=1}^{L}|F_{3,k}| \\ &= \prod_{k=1}^{N}\sqrt{1-\frac{\sin^2(2gt\omega_k)}{\cosh^2(\beta\omega_k)}}. \end{aligned}\tag{41}$$

From the Eqs. (40) and (41) one can find each decoherence factor $|F_k|$ is less than unity, which implies that in the large L limit, $|F_k(t)|$ will go to zero under some reasonable condition. Now, we make some further analysis by introducing a cutoff number K_c similar to the discussion in Ref. [27]. We define the partial product as

$$|F_1(t)|_c=\prod_{k>0}^{K_c}|F_{1,k}|\geq|F_1(t)|, \tag{42}$$

from which the corresponding partial sum $\ln|F_1(t)|_c\equiv-\sum_{k>0}^{K_c}|\ln F_{1,k}|$. We can do some heuristic analysis in some special conditions such as confining the energy spectrum in a region from zero to a nonzero value $\omega_k\in[0,\omega]$. When the cutoff number K_c is small enough, in a finite long time, with some proper g we can pick out some tiny ω_k to make $gt\omega_k$ begin a small one and achieve the approximation $\sin^2(gt\omega_k)\approx(gt\omega_k)^2$. At a finite temperature we can have

$$\begin{aligned}\ln|F_1(t)|_c &= \frac{1}{2}\sum_{k>0}^{K_c}\ln\left(1-\frac{\sin^2(gt\omega_k)}{\cosh^2(\beta\omega_k)}\right) \\ &\approx -\frac{1}{2}\left(\sum_{k>0}^{K_c}\frac{\omega_k^2}{\cosh^2(\beta\omega_k)}\right)g^2t^2 \\ &= -\gamma t^2 \end{aligned}\tag{43}$$

where

$$\gamma=\frac{1}{2}g^2\sum_{k}^{K_c}\frac{\omega_k^2}{\cosh^2(\beta\omega_k)}.$$

From Eqs. (42) and (43), we find that the decoherence factors decay in a Gaussian form with time, therefore from Eq. (25) it is apparent that the witness quantity $\mathcal{R}$ will vanish in a finite time. Also from (43), if the temperature is very low and quite nearly to $T=0$, γ approaches zero, and the decoherence factors quantity $\mathcal{R}$ will not decay with time. It implies that, in our system, the temperature greatly affect the dynamics of BE. It is a rough calculation in our analysis, nevertheless it gives us some constructive results.

2. Numerical results

If the frequency distribution are in the region $\omega_k\in[0,\omega]$, the decoherence factors displays a Gaussian decay, which is analytically studied in the former section. And in Ref. [28], Gaussian decay was shown numerically in various distributions of couplings. So, here, we consider the distributions such as $[50,50+\delta]$, a higher frequency region. In Fig. 6 (a), the larger coupling strength g makes a stronger oscillations of BE, which is different from the case of bosonic environment. Mathematically, we can understand that from Eq. (40) and (41), parameter g can change the frequency of the periodic function. The collapses and revivals of $\mathcal{R}$ is also observed here, and the revival amplitude decreases with time. Fig. 6 (b) is

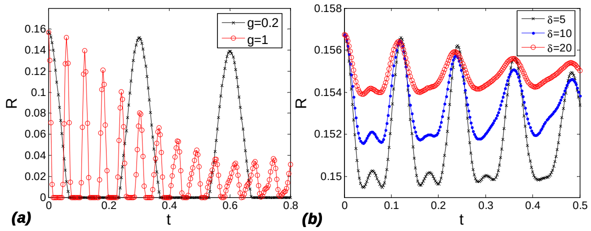

FIG. 6: **(a)** $\mathcal{R}$ versus time with different coupling parameter g. The couplings ω_k of the spins in environment is random in t nt is l 5. **(b)** on wid

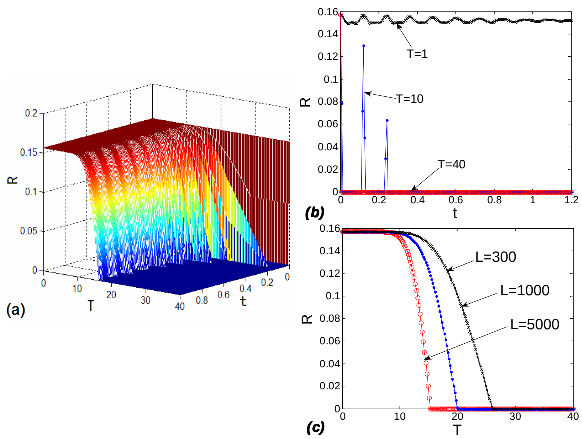

FIG. 7: **(a)** Three dimensional (3D) diagram of the $\mathcal{R}$ versus time and temperature, with $L = 300$, $g = 0.5$ and $\omega_k \in [50, 55]$. **(b)** Several part sections of the 3D diagram. Explicitly, it presents $\mathcal{R}$ versus time at different temperatures of $T = 1$, $T = 10$ and $T = 40$. **(c)** shows $\mathcal{R}$ decays with temperature at a fixed time $t = 0.005$ with different sizes of environment $L = 300, 1000$, and 5000.

a plot of $\mathcal{R}$ for different width of frequency distribution. The wider distribution will smear the collapse and revival phenomenon, namely, the BE evolve smoothly with time, and the BE is always there.

Effects of the thermal fluctuation on the dynamic of BE are shown in Fig. 7. The 3D plot (a) displays a flat at low temperatures about $T < 10$, which implies that the BE is stable against temperature in this region. When temperature is high enough, BE rapidly decreases to zero. In Fig. 7(b), we can see explicitly that temperature plays an important role in the dynamics of BE in this spin environment. It is just like a control process that when temperature is higher than some value, BE will decay sharply with time. At a very low temperature, from Eq. (40) and Eq. (41), we know that decoherence factors are approximately one with very small oscillations. And the BE is of a little change with time, namely, the BE is

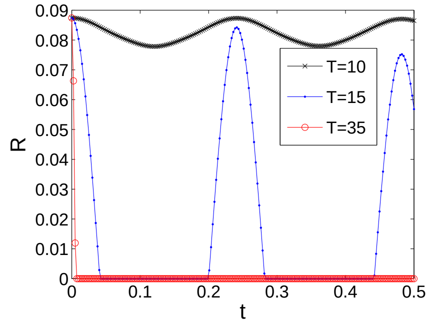

FIG. 8: $\mathcal{R}$ versus time at different temperatures, $T = 10$, $T = 15$ and $T = 35$. With the coupling $g = 0.5$, environment size $L = 300$ and $\omega_k \in [50, 55]$.

stable against time for low temperatures. This is a quite different property from the case of bosonic environment. In Fig. 7 (c), we also plot the BE against temperature for different size of environment. As we expected, the larger size of the spin environment accelerates the decaying process.

B. Second bound entangled state

We now consider the case that the two qutrits initially in the second bound entangled state (35) in the spin environment. We choose $\omega_k \in [50, 55]$ and study the finite-temperature effects on entanglement. In Fig. 8, we can see the similar phenomena with Fig. 6, namely, at low temperature BE only oscillates around the initial value of the BE. At higher temperature BE decays sharply with time, and revivals are also observed with revival amplitudes decreasing with time. We also study negativity, however, it always keeps zero which means there does not exist FE all the time.

IV. CONCLUSION

In summary, we have studied the decoherence phenomena of two-qutrit system couple to an enviroment when this open systems are initially prepared in a bound entangled state. The so-called "sudden death" phenomenon [5] of entanglement can be found in our studies as a common feature of BE evolution. Two typical pure dephasing systems, the bosonic and the spin systems are considered in order to show this kind of universality of decoherence process with BE. Here, we used the realignment criterion to characterize BE, and the PPT criterion to study FE. Beyond the negativity, we have introduced a novel witness quantity $\mathcal{R}$ to study BE. Those two approaches are operational and convenient to use. Two kinds of BE in initial state of the open system are considered, one is

given by Horodecki et al, and another is constructed from the unextendible product basis.

One of our central result is to express the quantity $\mathcal{R}$ and negativity $\mathcal{N}$ in terms of three decoherence factors, and these factors are analytically obtained. In the case of bosonic environments, the Gaussian decay and the exponential decay of the BE was found, and in the case of spin environment, we find that BE can display a Gaussian decay. In both environments, the collapse and revivals of the BE are observed for system frequency distribution being in a higher region with an appropriate width. Larger coupling strength g, larger environment size L and higher temperature will enhance the disentanglement process. For the Horocecki's BE in the spin environment, we find that the BE can be stable against low temperature increase.

Finally we have to point out that since we are lack of entanglement measure for two qutrits, the study of decoherence of entanglement here is incomplete. Nevertheless, the realignment criterion and the PPT criterion are very efficient to characterize BE. It will be interesting to consider decoherence of BE under other decoherence processes such as dissipation, and investigate the robustness of the BE.

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